By. Dr. Dharmendra Kr. Sharma
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Que 1 :- State and prove Theorem of Schwarz

Let F be a function of two variables defined over a certain neighbourhood N of (a,b)

if (i) F_x and F_y exist in N and

(ii) F_{xy} is continuous at (a,b) then F_{yx} exists at (a,b) and $F_{yx}(a,b) = F_{xy}(a,b)$ this follows in view of the fact that the continuity of (F_{xy}) implies the strong uniform existence of F_{xy} in view of the importance, we give a second independent proof of

Schwarz's theorem :- If (a,b) be a point in the domain of definition of a function $F(x,y)$ such that

(i) $F_x(x,y)$ and $F_y(x,y)$ exist in a certain neighbourhood N of (a,b) and (ii) one of $F_{xy}(x,y)$ and $F_{yx}(x,y)$ is continuous at (a,b) then the other exists and $F_{xy}(a,b)$ $= F_{yx}(a,b)$ i.e $\frac{\partial^2 F}{\partial y \partial x}(a,b) = \frac{\partial^2 F(a,b)}{\partial x \partial y}$

Proof :- Let us suppose according to the provisions of condition (ii) that $F_{xy}(x,y)$ is continuous at (a,b).

Let N be a neighbourhood of (a,b) in which $F_x(x,y)$ $F_y(x,y)$ and $F_{xy}(x,y)$ exist.

Let $(a+h, b+k)$ be any other point in N

Let $\Delta(h,k) = F(a+h, b+k) - F(a+h, b) - F(a, b+k) + F(a,b)$ ——— ①

and $g(x) = F(x, b+k) - F(x,b)$ ——— ②

$\therefore \quad \Delta(h,k) = g(a+h) - g(a)$

Now since F_x exists in N, $g(x)$ satisfies the conditions of Lagrange's mean value theorem in $[a, a+h]$ and hence there exists θ such that

$$g(a+h) - g(a) = h g'(a+\theta h) \quad \text{——— ③}$$

where $0 < \theta < 1$

$\therefore\ \Delta(h,k) = h\left[F_x(a+\theta h, b+k) - F_x(a+\theta h, b)\right]$

Again since F_{xy} exists in N

$F_x(a+\theta h, y)$ satisfies the conditions of Lagrange's mean value theorem and hence there exists θ such that

$$F_x(a+\theta h, b+k) - F(a+\theta h, b)$$

$$= k F_{xy}(a+\theta h, b+\theta k) \quad \text{--- (4)}$$

where $0 < \theta < 1$

$\therefore\ \Delta(h,k) = hk f_{xy}(a+\theta h, b+\theta h)$

& $\dfrac{\Delta(h,k)}{hk} = f_{xy}(a+\theta h, b+\theta' k)$

Hence $\lim\limits_{\substack{h\to 0\\ k\to 0}} \dfrac{\Delta(h,k)}{hk} = \lim\limits_{\substack{h\to 0\\ k\to 0}} F_{xy}(a+\theta h, b+\theta' k)$

$$= F_{xy}(a,b) \quad \text{--- (5)}$$

as $F_{xy}(x,y)$ is continuous at (a,b)

Again from (1) we have

$$\lim_{k\to 0} \frac{\Delta(h,k)}{hk} = \frac{1}{h}\lim_{k\to 0}\left[\frac{F(a+h, b+k) - F(a+h, b}{k} - \frac{F(a, b+k - F(a,b)}{k}\right]$$

$$= \frac{1}{h}\left[F_y(a+h, b) - F_y(a,b)\right] \quad \text{--- (6)}$$

Thus $\lim\limits_{k\to 0} \dfrac{\Delta(h,k)}{hk}$ and $\lim\limits_{\substack{h\to 0\\ k\to 0}} \dfrac{\Delta(h,k)}{hk}$ both exist.

Hence it follows the repeated limit

$\lim\limits_{h\to 0} \lim\limits_{k\to 0} \dfrac{\Delta(h,k)}{hk}$ exists and is equal to the double limit

$$\text{i.e. } \lim_{\substack{h\to 0\\ k\to 0}} \frac{\Delta(h,k)}{hk} = \lim_{h\to 0} \lim_{k\to 0} \frac{\Delta(h,k)}{hk} \quad \text{--- (7)}$$

Now from (6) it follows that

$$\lim_{h\to 0} \lim_{k\to 0} \frac{\Delta(h,k)}{hk} = \lim_{h\to 0} \frac{F_y(a+h, b) - F_y(a,b)}{h}$$

Hence $F_{yx}(a,b)$ exists and we have

$$\lim_{h\to 0} \lim_{k\to 0} \frac{\Delta(h,k)}{hk} = F_{yx}(a,b) \quad \text{--- (8)}$$

In view of (5), (7) and (8) we conclude that

$$F_{xy}(a,b) = F_{yx}(a,b)$$

$$\text{i.e. } \frac{\partial^2 F(a,b)}{\partial y\, \partial x} = \frac{\partial^2 F(a,b)}{\partial x\, \partial y}.$$